Conditional Value-at-Risk of the Completion Time in Fuzzy Activity Networks

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Abstract. The research deals with the evaluation of the Conditional Value-at-Risk (*CVaR*) for the completion time in scheduling problems represented as temporal activity networks where we assume that only a fuzzy representation for the activity integer valued durations is known to the scheduler. More precisely, we address the evaluation of the *CVaR* associated to a feasible schedule, and we extend the approach recently proposed for the case of interval valued durations. We develop and analyze a suitable computational method to obtain the fuzzy evaluation of the *CVaR* of the completion time of a given schedule. The proposed method enables to use the *CVaR* as quality criterion for wide classes of scheduling approaches considering risk-aversion in different practical contexts when only a fuzzy representation of activity durations is known.

Keywords: *CVaR*, Project Scheduling, Makespan, Fuzzy intervals, Activity Networks

1 Introduction

This paper addresses the evaluation of the Conditional Value-at-Risk (*CVaR*) of the completion time (hereinafter indicated as makespan) when the scheduling problem is represented by a temporal network (Elmaghraby 1977). At this aim, we consider scheduling models with a fixed and given set of precedence relations, but (independent) fuzzy processing times. More specifically, this research work assumes that non-deterministic durations are considered as fuzzy sets (Słowiński and Hapke 2000). The objective is to evaluate the *CVaR* of the makespan which is in general an NP-complete problem (Hagstrom 1988). We extend the approach recently proposed in (Meloni and Pranzo 2020), by developing and analyzing a suitable computational method to obtain the fuzzy evaluation of the *CVaR* of the makespan of a given schedule. The proposed algorithms are pseudo-polynomial in the general case, but they have been tested on a wide set of realistic instances for the experimental validation of their efficiency. The evaluation of the *CVaR* of the uncertain makespan $\mathbf{C}_{max}$ is an issue arising in several practical applications such as: project or task bidding, risk evaluation and mitigation, due date setting or acceptance, order proposal and acceptance, and robust scheduling (Elmaghraby 2005, Lawrence and Sewell 1997). The goals of this research is to propose and test a method for computing the *CVaR* of the makespan in the case of fuzzy durations, and to stimulate future research works devoted to the development of algorithms and/or other useful performance measures.

2 The *CVaR* of the makespan

In many applications the scheduler may be risk-averse and may prefer solutions that do not just perform well “on average", but that also perform satisfactorily “in most cases" (Elmaghraby 2005). Nevertheless there is no universally accepted single risk measure. In fact, each measure has its own advantages and disadvantages (Bertsimas *et. al.* 2004) and the choice also reflects a subjective preference of the decision makers. Several scalar performance indicators have been used to characterize the makespan $\mathbf{C}_{max}$ of a stochastic activity network. They include the *CVaR* at a probability level γ:
$CVaR_\gamma(\mathbf{C}_{max}) = E(\mathbf{C}_{max}|\mathbf{C}_{max} \geq q_\gamma(\mathbf{C}_{max}))$. It is also called γ-Tail Expectation or Expected shortfall (at level γ) of the makespan, and is related to the Value-at-Risk $VaR_\gamma(\mathbf{C}_{max})$ (Bertsimas *et. al.* 2004, Rockafeller 2007): $CVaR_\gamma(\mathbf{C}_{max}) = \frac{1}{1-\gamma}\int_\gamma^1 VaR_\beta(\mathbf{C}_{max})\,d\beta$.
Where $VaR_\gamma(\mathbf{C}_{max}) = inf\{t : prob(\mathbf{C}_{max} \leq t) \geq \gamma\}$ is a measure commonly used in finance which are gaining momentum also in scheduling (Lawrence and Sewell 1997, Sarin *et. al.* 2014). $VaR_\gamma(\mathbf{C}_{max})$ represents a threshold that is exceeded in $(1-\gamma)100\%$ of all cases, while the $CVaR_\gamma(\mathbf{C}_{max})$ represents the expected value of all cases exceeding the threshold $VaR_\gamma(\mathbf{C}_{max})$. Considering the definitions, the following holds for all $\gamma \in [0, 1]$: $VaR_\gamma(\mathbf{C}_{max}) \leq CVaR_\gamma(\mathbf{C}_{max})$. If a decision maker is not only concerned with the frequency of undesirable outcomes, but also with their severity, $CVaR_\gamma$ is recommended instead of VaR_γ (Bertsimas *et. al.* 2004, Sarin *et. al.* 2014). Higher values of γ are chosen by decision makers who are more risk-averse, and $\gamma = 0$ represents the risk-neutral choice. In fact, as γ tends to 1 (i.e., its upper extremum), the $CVaR_\gamma(\mathbf{C}_{max})$ tends to the worst case W; while when γ tends to 0, $CVaR_\gamma(\mathbf{C}_{max})$ tends to the expected value of the makespan $E(\mathbf{C}_{max})$. The features offered by *CVaR* are also useful in planning and scheduling problems based on stochastic activity networks models (Meloni and Pranzo 2020, Sarin *et. al.* 2014).

3 Fuzzy temporal activity networks

A *fuzzy temporal activity network* ($FTAN$) is a temporal activity nertwork with fuzzy valued durations. It can be defined by the pair $(G, \mathbf{D})$, where $G = (N, A)$ is the precedence DAG, the set of nodes N is associated to events, the set A of arcs represents the activities, and $\mathbf{D} = (\mathbf{D}_1, \dots, \mathbf{D}_m)$ is the vector of m fuzzy durations associated to the m arcs in A representing the activities. The network is directed, connected, and acyclic with single source and sink nodes. In a $FTAN$, all quantities that depend on the activity durations have a fuzzy characterization. Therefore, the starting and completion time of any activity, and the makespan $\mathbf{C}_{max}$ are all fuzzy quantities, i.e., fuzzy sets of the real line $\mathbb{R}$. A fuzzy set M of the universe of values X is characterized by a membership function μ_M which takes its value in interval $[0, 1]$. For each element $x \in X$, $\mu_M(x)$ defines the degree to which x belongs to $M = \{x \in X; \mu_M(x) \in [0, 1]\}$. An α-level cut (or α-cut, for short) of M is the crisp set $M_\alpha = \{x \in X|\mu_M(x) \geq \alpha\}$. The support of M is the crisp set $supp(M) = \{x \in X|\mu_M(x) > 0\}$. The duration of all the activities is a fuzzy number which is defined as a bounded support fuzzy quantity whose α-cuts are closed intervals. More specifically, we consider integer fuzzy numbers which are characterized as follows: *i)* the support is a closed integer interval denoted as $[\underline{M}, \overline{M}]$; *ii)* M is normal, i.e., there exists $\hat{x} \in [\underline{M}, \overline{M}]|\mu_M(\hat{x}) = 1$; *iii)* for any $x_1, x_2 \in [\underline{M}, \hat{x}]$, $\mu_M(x_1) \leq \mu_M(x_2)$ holds; and *iv)* for any $x_1, x_2 \in [\hat{x}, \overline{M},]$, $\mu_M(x_1) \geq \mu_M(x_2)$ holds. Given the fuzzy durations $\mathbf{D} = (\mathbf{D}_1, \dots, \mathbf{D}_m)$, the extension principle (Zadeh 1965) provides a powerful technique to extend a real continuous function of the activity durations (such as $CVaR_\gamma(\mathbf{C}_{max})$) to a fuzzy function $F(\mathbf{D})$ of the fuzzy durations $\mathbf{D}$. Moreover, the decomposition by α-cuts can be used to compute that fuzzy function by means of a decomposition method (Nguyen 1978): $[F(\mathbf{D})]_\alpha = F([\mathbf{D}_1]_\alpha, \ldots, [\mathbf{D}_m]_\alpha)$. According to this method, the membership function $\mu_F(x)$ of $F(\mathbf{D})$ can be reconstructed from its α-cuts F_α as follows: $\mu_F(x) = max\{\alpha : x \in F_\alpha\}$. We adopt a piecewise linear model for the membership function of the activity durations to simplify either information collection and computational aspects. We use a representation based on the full breakpoints ordered sequence which is very general and can be easily adapted to the cases of popular models such as triangular, trapezoid, and six-points approximated functions (Fortemps 1997).

4 Evaluation of the $CVaR$ of the makespan in *FTANs*

The case of crisp (i.e., ordinary) interval durations can be considered as a special case of *FTAN*. For this special case, in (Meloni and Pranzo 2020) an algorithmic approach has been proposed and experimentally validated. This approach is based on a counting approach to evaluate the $CVaR_\gamma$ for the makespan. The counting approach, starting from the pessimistic makespan (i.e., the worst possible makespan W), counts backwards how many configurations lead to each possible makespan value. This process is implemented as an iterative procedure which continues until sufficient information has been gathered to compute the $CVaR$ at a desired probability level γ.

On the basis of the computational results reported in (Meloni and Pranzo 2020), in this work we adopt an algorithm configuration for the crisp case which is able to determine a fast and extremely good estimation of $CVaR$ of the makespan in $O(\Gamma^2 m^2)$, where m is the number of arcs of the network, and Γ is the amount of uncertainty of the activity network represented by the size (in terms of number of integers) of the time interval $[\underline{C}_{max}, \overline{C}_{max}]$, where $\underline{C}_{max}$ ($\overline{C}_{max}$) is the makespan when all the activities durations are at their minimum (maximum) value. According to the α-cuts decomposition method, we follow an approximated approach for the evaluation of $CVaR_\gamma$ of the makespan for more general *FTANs*. To this aim, the basic algorithm for $CVaR_\gamma$ evaluation involving ordinary (i.e., non-fuzzy) intervals can be extended to solve the fuzzy cases, by the decomposition of the membership functions of the activity durations into a finite number of α-cuts. In the proposed method the basic algorithm can be applied on the instances associated to the selected α-levels to obtain the corresponding α-cuts of the desired fuzzy $CVaR_\gamma$ evaluation. This finite number of α-cuts are used to obtain an approximated reconstruction of the membership function $\mu_F(x)$ of $CVaR_\gamma$. Applying the general reconstruction rule described in the previous section (i.e., $\mu_F(x) = max\{\alpha : x \in F_\alpha\}$) to a finite set of α-cuts produces a stepped function that can be interpolated with a piece-wise linear function using the extreme points of the α-cuts as breakpoints. More specifically, in the α-cuts decomposition, for each selected level α, each fuzzy duration is cut at level α. This decomposition gives a set of activity networks with interval valued durations, each of which can be solved as a crisp instance. Then, in the successive fuzzy reconstruction procedure, an approximation of the fuzzy membership function of $CVaR_\gamma$ is determined from their α-cuts (e.g., see (Fargier *et. al.* 2000)).

This method is simple to implement but it could be intractable if ran for too many cuts and can be carried out only on a selection of suitably chosen level-cuts. Thus an issue comes from the selection of the relevant α-cuts. Possible solutions include: *i)* to choose α-cuts arbitrarily, e.g., according to a precision degree fixed by the user; *ii)* to use a given number of α-cuts at fixed α levels. Since we consider fuzzy durations represented by piece-wise linear functions, we adopt a more suitable choice of the levels α allowing for an accurate computation of the fuzzy quantities of interest. In fact, the resulting fuzzy quantities will be described by piece-wise linear functions too. Hence, the relevant α-cuts will be those corresponding to the breakpoints (i.e., of the right or left parts) of these fuzzy intervals.

Assuming these levels known (otherwise they can be easily determined in a pre-processing phase), an exact (approximate) interval-based procedure applied to the breakpoint values would compute the actual (approximate) fuzzy $CVaR$ values. In the proposed method, the α-cuts decomposition has a preliminary step devoted to determine the α-levels to adopt on the basis of either a specific input or default setting. We can then apply the algorithm to solve the case of interval valued durations to know the corresponding α-cuts of the $CVaR_\gamma$. The fuzzy $CVaR_\gamma$ is then reconstructed from its α-cuts and returned as output. Considering the number K of α-cuts used in the adopted decomposition, the overall complexity of the algorithm is $O(K\Gamma^2 m^2)$. In particular, associating the α-cuts to breakpoints (which is adopted as default scheme in our method) yields to a complexity $O(K_B \Gamma^2 m^2)$, where K_B is the overall number of the different α values in breakpoints in the durations contained in the $FTAN$.

A computational study is conducted to test the proposed approach on a benchmark set of realistic project scheduling problems. The overall speed and quality of the proposed method makes it an enabling tool for the use of $CVaR$ as an analysis criterion in fuzzy scheduling problems, while the trade-off between accuracy and computation effort indicates a possible research direction regarding the strategies for choosing the decomposition scheme in terms of both structure and size of the α sample set. Further research directions include the application of the proposed methodology in real contexts, and the improvement in the performance of the algorithm devoted to solve crisp-interval instances.

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