Ultimate Instance Reduction for the Routing Open Shop

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1 Introduction

In the routing open shop problem a fleet of mobile machines has to process a set of immovable jobs located at the nodes of some transportation network, described by an undirected edge-weighted graph $G = \langle V; E\rangle$, where each node contains at least one job, and weight $\mathrm{dist}(u, v)$ represents travel times between nodes u and v. Each machine M_i has to perform an operation O_{ji} on each job J_j in open shop environment, the processing times p_{ji} are given. All the machines start from the same node v_0 referred to as *the depot* and have to return to the depot after processing all the job. No restriction on the machines traveling are in order: any number of machines can travel over the same edge of the network simultaneously, machines are allowed to visit each node multiple times. However, machine has to reach a node prior to be able to process jobs located there. The goal is to minimize the makespan $R_{\max}$, *i.e.* the completion time of the last machine's activity (either traveling back to the depot or performing an operation on a job located at the depot). The problem is clearly a generalization of the metric traveling salesman problem and therefore is NP-hard in strong sense even for single machine. On the other hand, it generalizes the classical open shop problem, which is well-known to be NP-hard for the case of three and more machines, and is polynomially solvable for the two-machine case (Gonzalez T.F. and Sahni S. 1976). Surprisingly, the routing open shop is NP-hard even in the two-machine case on the transportation network consisting of at least two nodes (including the depot) (Averbakh I. *et. al.* 2006). We use notation $ROm||R_{\max}$ for the routing open shop with m machines. Optional notation $G = X$ in the second field is used in case we want to specify the structure of the transportation network, with X being the name of the structure (*e.g.* K_p or *tree*). A set of instances of the $ROm|G = X|R_{\max}$ problem is denoted by $\mathcal{I}_m^X$ (or $\mathcal{I}_m$ for a general case of unspecified X).

The routing open shop problem was introduced by Averbakh I. *et. al.* (2005). In our research we utilize the *standard lower bound* on the optimal makespan from the same paper:

$$\bar{R} = \max\left\{\ell_{\max} + T^*, \max_{v\in V}\left(d_{\max}(v) + 2\mathrm{dist}(v_0, v)\right)\right\}. \tag{1}$$

Here $\ell_{\max} = \max\limits_i \sum\limits_{j=1}^n p_{ji}$ is the maximum machine load, $d_{\max}(v) = \max\limits_{j\in\mathcal{J}(v)}\left(\sum\limits_{i=1}^m p_{ji}\right)$ is the maximum length of job from node v, with $\mathcal{J}(v)$ being the set of jobs located at v, while T^* is the TSP optimum on G. The problem under research is so-called *optima localization* and can be described as follows: how much (by what factor) can optimal makespan differ from the standard lower bound $\bar{R}$ for a given class of instances $\mathcal{K}$? More precisely, for some class $\mathcal{K}$ we want to find

$$\alpha(\mathcal{K}) = \sup_{I\in\mathcal{K}}\alpha(I) = \sup_{I\in\mathcal{K}}\frac{R^*_{\max}(I)}{\bar{R}(I)}.$$

Here $R^*_{\max}(I)$ and $\bar{R}(I)$ denote optimal makespan and the value of $\bar{R}$ for I, respectively, and $\alpha(I)$ is referred to as the *abnormality* of instance I.

It is known that for the classical two-machine open shop (which can be denoted as $RO2|G = K_1|R_{\max}$ for consistency) optimal makespan always coincides with the standard lower bound, therefore $\alpha\left(\mathcal{I}_2^{K_1}\right) = 1$ (Gonzalez T.F. and Sahni S. 1976). It is not the case for the three-machine problem, where optimal makespan can reach as much as $\frac{4}{3}\bar{R}$ (Sevastyanov S.V. and Tchernykh I.D. 1998). The value of $\alpha\left(\mathcal{I}_4^{K_1}\right)$ is still an open question, however we have no evidence that it is greater than $\frac{4}{3}$. Needless to say, that similar research for the routing open shop is probably harder even for $m = 2$, because the value $\alpha\left(\mathcal{I}_m^{K_p}\right)$ might depend both on m and p. However, it was recently established that $\alpha\left(\mathcal{I}_3^{K_2}\right) = \frac{4}{3}$ (Chernykh I. and Krivonogova O. 2020).

Current research for two machines up to the moment is as follows:

1. $\alpha\left(\mathcal{I}_2^{K_2}\right) = \frac{6}{5}$ (Averbakh I. *et. al.* 2005);
2. $\alpha\left(\mathcal{I}_2^{K_3}\right) = \frac{6}{5}$ (Chernykh I. and Lgotina E. 2016);
3. $\alpha\left(\mathcal{I}_2^{tree}\right) = \frac{6}{5}$ (Krivonogova O. and Chernykh I. 2019).

This paper addresses a natural question: how to stop this infinite series of incremental results and still reach an ultimate goal of discovering the general value $\alpha\left(\mathcal{I}_2\right)$.

2 Instance transformations

The research of some extremal (with respect to the standard lower bound) properties of the set of instances (such as optima localization) is often based on some instance transformation procedures. Suppose we have some transformation which obtains instance $\tilde{I}$ from I. Such a procedure is called *reversible* if any feasible schedule for $\tilde{I}$ can be treated as a feasible schedule for I. Reversibility means that $R^*_{\max}(\tilde{I}) \geqslant R^*_{\max}(I)$. The transformation $I \to \tilde{I}$ is referred to as *valid* if it preserves the standard lower bound: $\bar{R}(\tilde{I}) = \bar{R}(I)$. Obviously for any valid and reversible transformation $I \to \tilde{I}$ we have $\alpha(\tilde{I}) \geqslant \alpha(I)$. That observation serves as a foundation for the following approach to investigate the abnormality $\alpha(\mathcal{I})$ for some set of instances:

1. Describe a valid reversible transformation on $\mathcal{I}$ which *simplifies* the instance (*i.e.* reduces number of jobs to some constant, or simplifies the structure of the transportation network).
2. Describe the image $\tilde{\mathcal{I}}$ of $\mathcal{I}$ under that transformation. Find $\alpha(\tilde{\mathcal{I}})$.

There is a well-known transformation which reduces the number of jobs, referred to as *job aggregation* or *job grouping*. The idea is to combine a set of jobs into a single one adding up the processing times independently for each machine. Such a procedure was used, *e.g.*, in (Sevastyanov S.V. and Tchernykh I.D. 1998) for the classic open shop problem, and in (Chernykh I. and Lgotina E. 2016, Krivonogova O. and Chernykh I. 2019) for the two-machine routing open shop. While the procedure is clearly reversible, its validity has to be maintained explicitly. For example, it is possible to perform valid job aggregation for any instance of $Om||C_{\max}$ so that the resulting instance would contain at most $2m - 1$ jobs (Sevastyanov S.V. and Tchernykh I.D. 1998). As for $RO2||R_{\max}$, one can aggregate jobs in such a valid manner that every node (except for at most one) has a single job, and the "exceptional" one (if any) contains at most 3 jobs (Chernykh I. and Lgotina E. 2016). Such an exceptional node v is referred to as *overloaded*:

$$\Delta(v) = \sum_{j\in\mathcal{J}(v)} \sum_{i} p_{ji} > \bar{R} - 2\text{dist}(v_0, v).$$

However, it would be of the most interest to describe some valid reversible transformation to simplify the structure of G. An example of such a reduction is so-called *terminal edge contraction*, which can be described as follows. Suppose G contains a terminal node $v \neq v_0$ with a single job J_j in $\mathcal{J}(v)$. Let u be the node adjacent to v, and $\tau = \text{dist}(u, v)$. We translate the job J_j to the node u, increase its operations processing times by 2τ, and eliminate the obsolete node v. Such a transformation is reversible, as one can treat the processing of a new operation O_{ji} as a concatenation of traveling of M_i from u to v, processing of the initial operation and traveling back to u. It is proved in (Chernykh I. and Lgotina E. 2019) that for any instance $I \in \mathcal{I}_2$ one can perform a valid transformation $I \to \tilde{I}$ such that the transportation network in $\tilde{I}$ contains at most two terminal nodes. This helps to efficiently reduce any tree to a chain. On the other hand a graph might have a complex structure even without terminal edges. Below we describe a new approach to the instance reduction which allows to significantly simplify the structure of a transportation network preserving the standard lower bound $\bar{R}$.

Consider an instance $I \in \mathcal{I}_2$. Let $\Delta = \sum\limits_{i,j} p_{ji}$ be the *total load* if I. Note that (1) implies

$$\Delta \leqslant 2\ell_{\max} \leqslant 2(\bar{R} - T^*). \tag{2}$$

Let cycle σ be an optimal solution of the underlying TSP. Any edge $e \notin \sigma$ is referred to as *chord*. A chord e is referred to as *critical* if removing it from G increases the standard lower bound $\bar{R}$.

Lemma 1. *Any instance $I \in \mathcal{I}_2$ contains at most one critical chord, which is incident to the depot v_0.*

Proof. Note that the definition (1) does not depend on any distance between two non-depot nodes, therefore a chord may be critical only if it is incident to v_0. Suppose a chord $[v_0, v]$ is critical and τ is the new distance between v_0 and v after removing e from G. Then $\bar{R} < 2\tau + d_{\max}(v) \leqslant T^* + d_{\max}(v)$. Assume we have another critical chord $[v_0, u]$, therefore $\bar{R} < T^* + d_{\max}(u)$. Combining those two inequalities we obtain $2\bar{R} < d_{\max}(u) + d_{\max}(v) + 2T^* \leqslant \Delta + 2T^*$. Lemma is proved by contradiction with (2). □

Lemma 2. *Let $I \in \mathcal{I}_2$, node v is overloaded and chord $[v_0, u]$ is critical. Then $u = v$.*

Proof. We have $\Delta(v) > \bar{R} - 2\text{dist}(v_0, v) \geqslant \bar{R} - T^*$ and $d_{\max}(u) > \bar{R} - T^*$. Assume $u \neq v$, then $\Delta \geqslant \Delta(v) + d_{\max}(u) > 2(\bar{R} - T^*)$. Lemma is proved by contradiction with (2). □

Theorem 1. *Let $I \in \mathcal{I}_2$ such that the depot v_0 is overloaded. Then $\alpha(I) = 1$.*

Proof. Note that $\Delta(v_0) > \bar{R}$. It follows from Lemma 2 that I contains no critical chords, therefore eliminating all the chords is a valid (and reversible) transformation of I. Now let us replace all the jobs except the ones in the depot with a new single job J' with operations processing times $p'_i = T^* + \sum\limits_{J_j \notin \mathcal{J}(v_0)} p_{ji}$, and locate J' at v_0. Obsolete nodes (all except v_0) can now be removed from G, therefore G is transformed into a single-node graph and instance is reduced to the classic $O2||C_{\max}$ problem, for which we know that optimal makespan coincides with the standard lower bound. Such a transformation is reversible, as soon as we can treat the processing of operations of job J' as traveling along the optimal cycle and processing the jobs on the way. It is therefore sufficient to prove the validity of the transformation: $\sum_i p'_i = 2T^* + \Delta - \Delta(v_0) \leqslant 2T^* + 2(\bar{R} - T^*) - \Delta(v_0) < \bar{R}$. □

Now we describe a *chain contraction* transformation. Suppose G contains a chain $C = (v - v_1 - v_2 - \ldots - v_k - u)$, all the nodes $v_1, \ldots, v_k$ are of degree 2, and none of them is the depot. Let τ be the length of chain (the distance between v and u along C) and $\mathcal{J}(C)$ is the set of jobs from nodes $v_1, \ldots, v_k$. We now replace the subchain $v_1 - \ldots - v_k$ with a new *special* node v_C containing single job J_C with processing times $p_{Ci} = \tau + \sum\limits_{J_j \in \mathcal{J}(C)} p_{ji}$, and set weights of edges $[v, v_C]$ and $[v_C, u]$ to zero.

Such a transformation is not reversible in general. To make it reversible we need to apply certain restriction on schedules for the transformed instance:

1. If machine arrives at J_C from one end (say, from v), the machine is considered to be at another end (say, u) after the completion of operation of job J_C.
2. Any machine can bypass the node v_C, but this takes τ time units.

We say that the chain contraction transformation is *conditionally reversible*, meaning that we obtain a special node which has to be treated as described above.

The main result of this paper is the following

Theorem 2. *For any instance $I \in \mathcal{I}_m$ there exists a combination of valid chord eliminations and chain contractions $I \to \tilde{I}$ such that $\tilde{I}$ contains at most $2m$ nodes from which at most m are special.*

Moreover, the structure of the resulting instance $\tilde{I}$ is not arbitrary. For instance, for $m = 2$ the most general structure we need to investigate is the cycle $(v_0 - v_1 - v_2 - v_3 - v_0)$ with additional chord $[v_0, v_2]$ and v_1, v_3 being special nodes. Our working conjecture is that for any instance I of such a special structure $\alpha(I) = \frac{6}{5}$, and therefore $\alpha(\mathcal{I}_2) = \frac{6}{5}$. Theorem 2 can still be useful for a general $ROm||R_{\max}$ problem, although the research for the tight optima localization interval for $m \geqslant 3$ is difficult even for the classic $Om||C_{\max}$ problem.

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