

# Minimizing Flow Time on a Single Machine with Job Families and Setup Times.

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**Context and problem description** In (Mason and Anderson 1991), the authors consider a one machine scheduling problem with the goal of minimizing the average weighted flow time. In this problem, jobs are grouped into families with the property that a setup time is required only when processing switches from jobs of one family to jobs of another family. Furthermore, setup times are additive: each setup for a new family of job consists of the setdown from the previous family followed by a setup of the new family. These additive setups are considered as sequence-independent. Note that a setup is required at time 0 before the very first job is processed. For this problem, they define several properties of an optimal solution. These properties are then used in a branch-and-bound procedure.

If jobs belonging to the same family have equal processing times and unit weights, the application of their results directly leads to a polynomial time algorithm. However, if no setup time is required at time 0, i.e. before the very first job, their results cannot be applied directly. The first goal of this paper is to prove this affirmation. The second objective is to show how their results can be adapted in this special case. These results are then used to define a polynomial time algorithm to compute the optimal flow time for this special case. More generally, the objective is to use this algorithm for solving a parallel machine scheduling problem with time constraints and machine qualifications described in (Malapert and Nattaf 2019).

The problem considered in this paper is now formally described. Consider the problem of scheduling a set  $\mathcal{N}$  of  $n$  jobs on one machine. Each job belongs to a family  $f \in \mathcal{F}$  and the family associated with a job  $j$  is denoted by  $f_j$ . Each family is then associated with a number of jobs to process  $n_f$ , a processing time  $p_f$ , and a sequence independent setup time  $s_f$ . Thus, switching the production from family  $f$  to  $f' \neq f$  will require a setup time  $s_{f'}$ . Note that no setup time is needed between two jobs of the same family. The important difference to the original problem is that the setup times are not additive anymore because no setup time is required at time 0 before the very first job is processed. The goal is to minimize the flow time which is the sum of the finishing time of all jobs.

**Optimal solution properties** The goal of this section is to describe several properties and characteristics of an optimal solution. The properties will then be used to design our polynomial-time algorithm.

To represent a solution, let  $S$  be a sequence representing an ordered set of  $n$  jobs. Then,  $S$  can be seen as a series of blocks, where a block is a maximal consecutive subsequence of jobs in  $S$  from the same family. Let  $B_i$  be the  $i$ -th block of the sequence  $S = \{B_1, B_2, \dots, B_r\}$ .

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Successive blocks contain jobs from different families. Therefore, there will be a setup time before each block (except the first one). The idea is to adapt SPT Smith's rule (Smith 1956) for blocks instead of individual jobs. To this end, blocks are considered as individual jobs with processing time  $P_i$  and weight  $W_i$  with:  $P_i = s_{f_i} + |B_i| \cdot p_{f_i}$  and  $W_i = |B_i|$  where  $f_i$  denotes the family of jobs in  $B_i$  (which is the same for all jobs in  $B_i$ ).

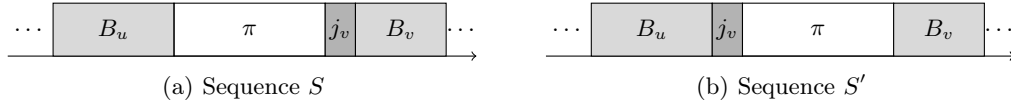
The following theorem states that there exists an optimal solution  $S$  containing exactly  $|\mathcal{F}|$  blocks and that each block  $B_i$  contains all jobs of the family  $f_i$ .

**Theorem 1.** *Let  $\mathcal{I}$  be an instance of the problem. There exists an optimal solution  $S^* = \{B_1, \dots, B_{|\mathcal{F}|}\}$  such that  $|B_i| = n_{f_i}$  where  $f_i$  is the family of jobs in  $B_i$ .*

*Proof.* Consider an optimal solution  $S = \{B_1, \dots, B_u, \dots, B_v, \dots, B_r\}$  with two blocks  $B_u$  and  $B_v$  ( $u < v$ ), containing jobs of the same family  $f_u = f_v = f$ . We show that moving the first job of  $B_v$  at the end of block  $B_u$  can only improve the solution.

Let us define  $P$  and  $W$  as:  $P = \sum_{i=u+1}^{v-1} P_i + s_f$  and  $W = \sum_{i=u+1}^{v-1} |B_i|$ . Note that  $P$  (resp.  $W$ ) is the total time of all jobs, including setups, (resp. the number of jobs) performed between the last job of  $B_u$  and the first job of  $B_v$ . Let us call  $\pi$  this partial sequence.

Let  $S'$  be the sequence formed by moving the first job of  $B_v$ , say job  $j_v$ , at the end of block  $B_u$ , that is swapping the position of  $\pi$  and  $j_v$  (see Fig. 1).



**Fig. 1.** Construction of sequence  $S'$  from sequence  $S$ .

First, the difference on the flow time is computed in the cases where  $|B_v| > 1$  and where  $|B_v| = 1$ .

If  $|B_v| > 1$  (that is there are still jobs left in  $B_v$  after the removal of job  $j_v$ ), then the flow times of jobs sequenced before  $B_u$  or after  $B_v$  do not change. All the jobs in  $\pi$  have their flow times increased by  $p_f$  and  $j_v$  has its flow time decreased by  $P$ , giving a difference in the total flow time of:

$$FT_{S'} - FT_S = W \cdot p_f - P$$

If  $|B_v| = 1$ , then  $B_v$  is left with no jobs after moving job  $j_v$ , and so the setup time associated with  $B_v$  is deleted from the sequence. This reduces the flow times of all jobs sequenced after  $\pi$  by the amount of  $s_f$ , giving an additional reduction in the total net flow time:

$$FT_{S'} - FT_S = W \cdot p_f - P - \sum_{i=v+1}^r |B_i| \cdot s_f$$

Hence, if one can prove that  $P/W \geq p_f$ ,  $FT_S - FT_{S'} < 0$  and the flow time can only be improved in  $S'$ .

**Lemma 1.**  $\frac{P}{W} \geq p_f$

*Proof.* Consider the sequence  $S$  and suppose that  $p_f > P/W$ . Let  $S''$  be the sequence formed by moving the last job of  $B_u$  at the beginning of block  $B_v$ . If  $|B_u| > 1$ , then the difference of flow time is:  $FT_{S''} - FT_S = P - W \cdot p_f$ . Since, by definition,  $p_f > P/W$ , we have that  $FT_{S''} - FT_S < 0$  which contradict the fact that  $S$  is optimal.

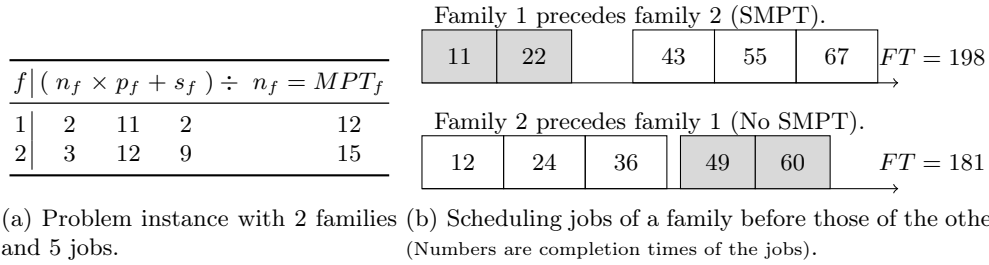
If  $|B_u| = 1$ ,  $FT_{S''} - FT_S = \begin{cases} W \cdot p_f - P - \sum_{i=v+1}^r |B_i| \cdot s_f & \text{if } u \neq 1 \\ W \cdot p_f - P - \sum_{i=v+1}^r |B_i| \cdot s_f - \sum_{i=u+1}^r |B_i| \cdot s_{f_{u+1}} & \text{if } u = 1 \end{cases}$   
 And then,  $S''$  is a better solution than  $S$  which is a contradiction. Hence, we have  $P/W \geq p_f$ .  $\square$

Therefore, by Lemma 1,  $FT_{S'} - FT_S \leq 0$ . Hence, swapping the position of job  $j_v$  with  $\pi$  leads to a solution  $S'$  at least as good as  $S$ . Repeated applications of this operation yield the result.

The mean processing time of a block  $B_i$  can be defined as  $MPT(B_i) = P_i/W_i$ . Theorem 2 generalized the SPT rule for blocks.

**Theorem 2.** *In an optimal sequence of the problem, the blocks 2 to  $|\mathcal{F}|$  are ordered by SMPT (Shortest Mean Processing Time). That is, if  $1 < i < j$  then  $MPT(B_i) \leq MPT(B_j)$ .*

This theorem is not exactly the same as the one in (Mason and Anderson 1991). Indeed, their theorem allows the ordering of all blocks according the SMPT rule while Theorem 2 orders only blocks 2 to  $|\mathcal{F}|$ . Figure 2 gives a counterexample showing that the SMPT rule is not optimal when there is no setup at time 0.



**Fig. 2.** The SMPT rule may lead to suboptimal solutions when no setup time is required at time 0.

*Proof (Th. 2).* Let  $S = \{B_1, B_2, \dots, B_u, B_{u+1}, \dots, B_{|\mathcal{F}|}\}$  be a sequence in which blocks  $B_u$  and  $B_{u+1}$ ,  $1 < u \leq u+1 \leq |\mathcal{F}|$ , are not in SMPT order; that is,  $MPT(B_u) > MPT(B_{u+1})$ . Consider the change in total flow time if the processing order of  $B_u$  and  $B_{u+1}$  is reversed. Clearly the flow times of any jobs originally scheduled before run  $B_u$  will not be changed. Also, the total time to complete blocks  $B_u$  and  $B_{u+1}$  will not be changed, so the flow times of any jobs scheduled after  $B_{u+1}$  will not be altered. Hence, only the flow times of the jobs in blocks  $B_u$  and  $B_{u+1}$  needs to be considered. Each job in  $B_{u+1}$  has its completion time reduced by  $P_u$  and each job in  $B_u$  has its completion time increased by  $P_{u+1}$ . Thus, the change in weighted flow time is given by:

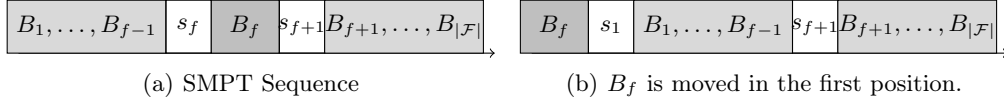
$$\Delta F_w = W_u \cdot P_{u+1} - W_{u+1} \cdot P_u$$

But, since  $MPT(B_u) > MPT(B_{u+1})$ ,  $P_u \cdot W_{u+1} > P_{u+1} \cdot W_u$ , and so  $\Delta F_w < 0$ .

The following section explains how these results are used to define a polynomial time algorithm for solving the problem.

**Polynomial-time algorithm** Theorem 1 states that there exists an optimal solution  $S$  containing exactly  $|\mathcal{F}|$  blocks and that each block  $B_i$  contains all jobs of family  $f_i$ . Theorem 2 states that the blocks  $B_2$  to  $B_{|\mathcal{F}|}$  are ordered by *SMPT*. Finally, one only needs to determine which family is processed in the very first block.

Algorithm 1 take as input the jobs grouped in blocks and in *SMPT* orders. The algorithm starts by computing the flow time of this schedule. Each block is then successively moved to the first position (see Figure 3) and the new flow time is computed. The solution returned by the algorithm is therefore the one achieving the best flow time.



**Fig. 3.** SMPT Sequence and Move Operation.

For sake of readability, let  $F(f)$  denote the internal flow time of the block  $f$ , *i.e* the flow time of its jobs when starting at time 0:  $\sum_1^{n_f} i \times p_f$

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**Algorithm 1:** SMPT Scheduling without setup at time 0.

**Data:**  $n_f, p_f, s_f$  for  $f \in \mathcal{F}$  in *SMPT* order ( $MPT_f \leq MPT_{f+1}$ ) such that  $n_f > 0$ .  
**Result:** The optimal flow time  $FT$ .

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// Compute the flow time  $FT_1$  of the SMPT sequence  $\{B_1, \dots, B_{|\mathcal{F}|}\}$ 
 $FT_1 \leftarrow F(1); \quad P \leftarrow n_1 \times p_1;$ 
for  $f \leftarrow 2$  to  $|\mathcal{F}|$  do
    // shift jobs of  $B_f$  and add their internal flow times
     $FT_1 \leftarrow FT_1 + n_f \times (P + s_f) + F(f)$ 
     $P \leftarrow P + s_f + (n_f \times p_f)$  // Ending time of  $B_f$ 

// Note that  $P$  is the makespan of the SMPT sequence
// Compute the flow time when the block  $B_f$  is in the first position
 $FT \leftarrow FT_1; \quad W \leftarrow n;$ 
for  $f \leftarrow |\mathcal{F}|$  to 2 do
     $W \leftarrow W - n_f$  // Number of jobs in  $\{B_1, \dots, B_{f-1}\}$ 
     $P \leftarrow P - s_f - (n_f \times p_f)$  // Ending time of  $B_{f-1}$  in the SMPT sequence
    // Compute the variations of the flow time
     $\Delta_f \leftarrow -(P + s_f) \times n_f$  // For  $B_f$ 
     $\Delta^- \leftarrow (n_f \times p_f + s_1) \times W$  // For  $\{B_1, \dots, B_{f-1}\}$ 
     $\Delta^+ \leftarrow (s_1 - s_f) \times (n - n_f - W)$  // For  $\{B_{f+1}, \dots, B_{|\mathcal{F}|}\}$ 
     $FT_f \leftarrow FT_1 + \Delta^- + \Delta_f + \Delta^+$  // For the entire sequence
    if  $FT_f < FT$  then  $FT \leftarrow FT_f$  // Update the best flow time

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The complexity for ordering the families in *SMPT* order is  $O(|\mathcal{F}| \log |\mathcal{F}|)$ . The complexity of Algorithm 1 is  $O(|\mathcal{F}|)$ . So, the complexity for finding the optimal flow time is  $O(|\mathcal{F}| \log |\mathcal{F}|)$ .

## References

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