Optima Localization for the Routing Open Shop: Computer-aided Proof

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1 Introduction

In the open shop problem sets of jobs $\mathcal{J} = \{J_1, \ldots, J_n\}$ and machines $\mathcal{M} = \{M_1, \ldots, M_m\}$ are given. Each of job J_j has to be processed by each machine M_i in arbitrary order, and this operation takes a given processing time p_{ji}. The goal is to minimize *the makespan* $C_{\max}$ which is defined as a maximum completion time of the operation. We use notation $Om||C_{\max}$ for the problem with m machines. It is known (Gonzalez T.F. and Sahni S. 1976) to be polynomially solvable in the case $m = 2$ and NP-hard for $m \geq 3$.

We consider the routing open shop problem being a generalization of the metric TSP and the open shop problem. Routing open shop, introduced in (Averbakh I. *et. al.* 2006, Averbakh I. *et. al.* 2005), can be described as follows. Jobs are located at the nodes of a transportation network described by an edge-weighted graph $G = \langle V; E\rangle$, each node contains at least one job. The weight $\mathrm{dist}(u, v)$ represents the travel time of any machine between those nodes. Mobile machines are initially located at some predefined node $v_0 \in V$ referred to as *the depot*. All the machines have to travel between nodes to process jobs in an openshop-like environment, and to return to the depot after completion of all the operations. The makespan $R_{\max}$ is the *return time* moment of the last machine after completion of all its operations, and has to be minimized. We denote this problem as $ROm||R_{\max}$, or as $ROm|G = X|R_{\max}$ in the case we want to specify the structure of the transportation network. Problem is known to be NP-hard even in trivial cases with single machine (equivalent to the metric TSP) and with two machines and just two nodes of the network (including the depot)(Averbakh I. *et. al.* 2006). The latter case is denoted as $RO2|G = K_2|R_{\max}$.

Consider the following *standard lower bound* on the optimal makespan, proposed in Averbakh I. *et. al.* (2005):

$$\bar{R} = \max\left\{\ell_{\max} + T^*, \max_{v\in V}(d_{\max}(v) + 2\mathrm{dist}(v_0, v))\right\}.$$

Here $\ell_{\max} = \max_i \sum_{j=1}^{n} p_{ji}$ is the maximum machine load, $d_{\max}(v) = \max_{j\in\mathcal{J}(v)} d_j = \max_{j\in\mathcal{J}(v)} \left(\sum_{i=1}^{m} p_{ji}\right)$ is the maximum length of job from node v, with $\mathcal{J}(v)$ being the set of jobs located at v. T^* denotes the TSP optimum on G with distance function $\mathrm{dist}(u, v)$.

One of the directions of the research of an NP-hard optimization problem is *optima localization*, i.e. the search of tight upper bound on the optimum in terms of the lower bound LB. More precise, the tight optima localization interval is an interval of type $[LB, \rho LB]$

with the smallest possible value of ρ guaranteed to contain an optimum value for any problem instance from a given set. The first tight optima localization interval for scheduling problems was found for $O3||C_{\max}$ in (Sevastyanov S.V. and Tchernykh I.D. 1998). This research required massive computer-aided enumeration based on the branch-and-bound method.

For the routing open shop problem this question was partly studied for the case of two machines. It is proved in Averbakh I. *et. al.* (2005) that optimum of any instance of $RO2|G = K_2|R_{\max}$ belongs to an interval $[\bar{R}, \frac{6}{5}\bar{R}]$, and the bounds are tight. Lately this result was generalized for the $RO2|G = K_3|R_{\max}$ (Chernykh I. and Lgotina E. 2016) and $RO2|G = tree|R_{\max}$ problems (Krivonogova O. and Chernykh I. 2019). Optima localization for the problem with three or more machines is still an open question even in case $G = K_2$.

2 Instance simplification operations

The research of the optima localization for the two-machine case is based on an instance reduction procedure which uses two simplification operations: job aggregation and terminal edge contraction.

Job aggregation operation (also known as *grouping*) utilizes a simple idea of replacing a number of jobs from the same node with a single aggregated job for which processing times equal to the total processing time of the respective operations combined. We use job aggregation to simplify the instance preserving the standard lower bound $\bar{R}$. A natural question arises, is it possible to perform a job aggregation of a whole set of jobs at some nodes. To answer that question, we use the following definition.

Definition 1. *A node v from $G(I)$ of some problem instance I is* overloaded *if*

$$\varDelta(v) = \sum_{J_j \in \mathcal{J}(v)} d_j > \bar{R} - 2\mathrm{dist}(v_0, v).$$

Otherwise the node is referred to as underloaded*.*

The job aggregation of the whole set of jobs in node v preserves $\bar{R}$ if and only if the node v is underloaded.

Another operation, terminal edge contraction, is based on the following idea: translate a single job from a terminal node v to an adjacent one u, modifying processing times of all of its operations to include travel times (back and forth) between v and u.

Again, we want to perform an edge contraction operation only if it does not lead to the growth of the standard lower bound $\bar{R}$. Otherwise, the edge is called overloaded. The following definition describes the exact condition, under which an edge is overloaded.

Definition 2. *Let $v \neq v_0$ is a terminal node in G and there is a single job $J_j \in \mathcal{J}(v)$. Let $e = [u, v]$ be an edge incident to v. Then edge e is* overloaded *if*

$$d_j + 2m\mathrm{dist}(u, v) + 2\mathrm{dist}(v_0, u) > \bar{R},$$

and is underloaded *otherwise.*

Overloaded elements make the instance somehow problematic. Fortunately, the number of such elements is rather small.

Lemma 1. *Any instance of the $ROm||R_{\max}$ problem contains at most $m - 1$ overloaded elements.*

Moreover, the number of jobs in the simplified instance is small. One of the main results of our research is the following

Lemma 2. *Let I be an instance of the $ROm||R_{\max}$ problem and any job aggregation in I leads to the growth of $\bar{R}$. Then every underloaded node in I contains exactly one job, and all the overloaded nodes (if any) contain at most $2m-1$ jobs altogether.*

Instance simplification preserving the lower bound allows one to reduce the search for the tight optima localization interval to the case with small number of jobs (depending on m) and with simpler structure of the transportation network. The next section covers the first attempts to discover optima localization interval for the three-machine routing open shop.

3 Optima localization for $RO3|G = K_2|R_{\max}$

For any instance of $RO3|G = K_2|R_{\max}$ we use v to denote the node other than the depot.

Back in 1998 Sevastyanov and Chernykh used a computer program to prove that for any instance of $O3||C_{\max}$ (which is equivalent to $RO3|G = K_1|R_{\max}$) optimal makespan does not exceed $\frac{4}{3}$ times standard lower bound. The program is based on an intelligent branch-and-bound-style enumeration of subsets of instances (with infinite cardinality). For each subset a *critical instance* with the greatest ratio of upper and lower bounds of the optimal makespan was found with a help of linear programming. The proof follows from the facts that the enumeration is complete (union of the subsets considered coincides with the whole sets of instances), and for each critical instance found the upper bound is within the range of $\frac{4}{3}\bar{R}$. It took about 200 hours of running time to complete the proof, including building the structure of subsets and the search for critical instances for each one. As it was clear that a direct application of the same approach would take enormous amount of time, we focused our research on the possibilities to make the proof-building process more efficient. As a result, we were able to complete the research of the optima localization of the $RO3|G = K_2|R_{\max}$ problem and to prove the following theorem constructively.

Theorem 1. *For any instance of the $RO3|G = K_2|R_{\max}$ problem there exists a feasible schedule S such that $R_{\max}(S) \leq \frac{4}{3}\bar{R}$.*

One part of the proof is based on a description of a set of sufficient conditions which allow to reduce an instance to the case of $O3||C_{\max}$. Another one used the computer-aided approach with some fine-tuning applied. As a result, the proof-building process was complete in about 28 hours.

Let us focus on the running time reduction techniques. First idea was to try to reduce the set of instances as much as possible without loss of generality. This is done by means of the following two lemmas.

Lemma 3. *For any instance of $RO3|G = K_2|R_{\max}$ with underloaded node v and $p_{\max} = \max p_{ji} \geqslant \frac{2}{3}\bar{R}$ the optimal makespan does not exceed $\frac{4}{3}\bar{R}$.*

Lemma 4. *Let I be an instance for $RO3|G = K_2|R_{\max}$ problem such that $\Delta(v_0) > 2\bar{R}$. Then $R^*_{\max} \leqslant \frac{4}{3}\bar{R}$.*

The influence of the application of different combinations of these restrictions on the running time for one of the special cases of the problem is presented in Table 1.

As one can observe, that influence is not that noticeable. Luckily, we discovered another reserve which surprisingly allowed one to reduce running time significantly.

Second idea was to reduce the set of instances by using symmetries induced by different enumerations of jobs and machines.

	$\Delta(v_0) \leq 2\bar{R}$	$\Delta(v_0)$ is arbitrary
$p_{\max} \leq \frac{2}{3}\bar{R}$	17:52 min.	19:27 min.
$p_{\max}$ is arbitrary	20:27 min.	29:31 min.

Table 1. Running time of the original program depending on the restriction applied.

	$\Delta(v_0) \leq 2\bar{R}$	$\Delta(v_0)$ is arbitrary
$p_{\max} \leq \frac{2}{3}\bar{R}$	00:10 min.	01:45 min.
$p_{\max}$ is arbitrary	00:09 min.	02:14 min.

Table 2. Running time of the modified program depending on the restriction applied.

Further ways to improve efficiency are based on details of the computer-aided approach and cannot be fully disclosed in the format of the current abstract. The results are covered in Table 2.

Thus we were able to reduce the running time (for one of the special cases) by the factor of almost 200, which gives us hope that the computer-aided approach can still be used for wider classes of problems, *i.e.* $O4||C_{\max}$ (an intriguing case, as we no evidence that the optimal makespan can be greater than $\frac{4}{3}\bar{R}$), $RO3|G = K_3|R_{\max}$ and so on.

4 Conclusion

The main results of this paper are the following.

1. Description of the extremal properties of overloaded elements of $ROm||R_{\max}$ problem.
2. The optima localization of the special case of the $RO3|G = K_2|R_{\max}$ problem.
3. Developments of the computer-aided approach with a significant reduction of the running time.

An intriguing open question from (Sevastyanov S.V. and Tchernykh I.D. 1998) still remains: does there exist an analytic proof of Theorem 1 (as well as the optima localization result for $O3||C_{\max}$), such that doesn't require any computer-aided enumeration.

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