On the Activity Criticality in Project Scheduling with Generalized Precedence Relationships

Lucio Bianco, Massimiliano Caramia and Stefano Giordani

University of Rome "Tor Vergata", Italy
bianco, caramia, giordani@dii.uniroma2.it

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1 Introduction

It is well known that a project network with Generalized Precedence Relations (GPRs), due to the presence of maximum and minimum times lags, may contain cycles and a critical path may contain cycles of zero length (De Reyck, 1998). Consequently, it may happen that the project duration increases when the duration of a critical activity is shortened. This phenomenon was firstly studied by Wiest (1981). In another seminal paper, Elmaghraby and Kamburowski (1992) further studied this anomaly under GPRs. They introduced five different criticality types (i.e., start-critical, finish-critical, backward-critical, forward-critical, and bi-critical) and the new concept of flexibility. Later, De Reyck (1998) in his doctoral thesis, revisited these concepts adapting them to an Activity on Nodes (AON) representation of the project network and proposed a method for recognizing criticalities and inflexibilities of an activity, based on the types of its ingoing and outgoing precedence relations. Since the Ph.D. work of De Reyck appears not to be present in the open literature, the reader may find the analysis of De Reyck in the book by Demeulemeester and Herroelen (2002). To the best of our knowledge, the analysis made by De Reyck has been widely accepted and not revised by any point of view in the last twenty years.

In this work, starting from some concerns related to the criticality definitions of the activities and potential failures of De Reyck's method for the analysis of activity criticalities and flexibilities, we propose a new method. Section 2 provides some definitions and notations. In Section 3, by means of one example, we show some potential failures of the De Reyck's method, giving in Section 4 new results. Similar remarks can also be made for the method proposed by Elmaghraby and Kamburowski (1992). In Section 5 we provide a brief outline of the new method after having redefined and discussed the types of criticalities.

2 Definitions and notations

Accordingly to De Reyck (1998) and to Demeulemeester and Herroelen (2002), hereafter we assume that a project is modeled by means of an AON network $N = (V, A; d, \delta)$. V is the set of nodes, with $V = V^r \cup \{1, n\}$, where $V^r = \{2, \dots, n-1\}$ is the set of $n-2$ real activities, d_i is the duration of activity $i \in V^r$, and nodes 1 and n are two additional dummy activities, with duration equal to zero, representing project beginning and completion, respectively; without loss of generality, we assume the real activity durations being integers and positive. A is the set of arcs representing GPRs between pairs of activities. An arc may model a start-to-start (SS), a start-to-finish (SF), a finish-to-start (FS) and a finish-to-finish (FF) precedence relation with minimum or maximum time lags for an overall number of eight relations that may be represented, i.e., $SS_{ij}^{min}(\delta)$, $SS_{ij}^{max}(\delta)$, $SF_{ij}^{min}(\delta)$, $SF_{ij}^{max}(\delta)$, $FS_{ij}^{min}(\delta)$, $FS_{ij}^{max}(\delta)$, $FF_{ij}^{min}(\delta)$, $FF_{ij}^{max}(\delta)$, where δ is the minimum or maximum time lag. It is well known (see e.g., Demeulemeester and Herroelen, 2002) that a GPR with maximum time lag is equivalent to a GPR with minimum time lag with opposite direction and opposite time lag, i.e., $SS_{ij}^{max}(\delta) \equiv SS_{ji}^{min}(-\delta)$, $SF_{ij}^{max}(\delta) \equiv FS_{ji}^{min}(-\delta)$, $FS_{ij}^{max}(\delta) \equiv SF_{ji}^{min}(-\delta)$, and $FF_{ij}^{max}(\delta) \equiv FF_{ji}^{min}(-\delta)$. Hence, we can always model the project activities and their relationships with a GPRs AON network with only minimum time lag, but the resulting network may contain cycles, due to maximum time lag. Therefore, without loss of generality, in the following we assume that the project has only GPRs with minimum time lags.

It is well known that with the transformations of Bartush *et al.* (1988) we can represent the project network in a so called *standardized* form where there are, for example, only GPRs of type $SS_{ij}^{min}(\ell)$. This standardized network allows to calculate the project duration as the length of the longest path form node 1 to node n, i.e., the length of a critical path. Moreover, it allows to determine the critical activities and the critical arcs (i.e., critical precedences among critical activities).

3 An example showing potential failures

We show a project network example for which the method proposed by De Reyck fails in determining some activities' criticalities. Analog failures can also be shown on the method by Elmaghraby and Kamburowski. Let us consider the project network with GPRs in Figure 1a, with node weights being activities' durations. The standardized network (with only $SS_{ij}^{min}(\ell)$ precedences) is shown in Figure 1b, with arcs' weights being time lags. Let us consider the critical path $(1,2,3,4,5,4,6,7)$ with length equal to 20 (note that the path contains a cycle of length equal to zero). All the activities belong to the critical path. The criticalities according to De Reyck's method (see the definitions and Table 6 at page 124 of the book of Demeulemeester and Herroelen, 2002) are: activity 2, forward-critical; activity 3, finish-critical; activity 4, bi-critical; activity 5, start-critical; activity 6, finish-critical.

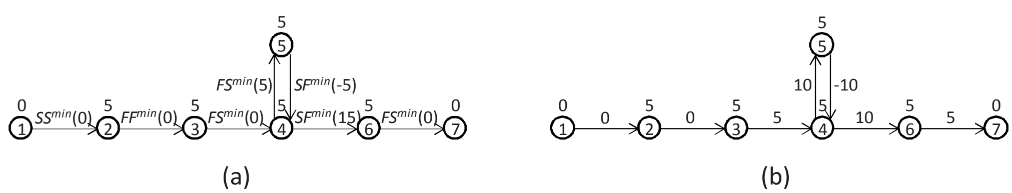

Fig. 1. The project network with GPRs of the example (a) and its standardized network (b)

Actually, the criticalities of activities 3, 4, and 5 are different, as the flexibility analysis reveals. Indeed, if it was $d_3 = 6$, it would be $\ell_{23} = -1$ and $\ell_{34} = 6$, with ℓ_{ij} being the length of arc (i,j) in the standardized network. In this case, the length of the longest path from node 1 to node 3 would be negative and hence we need to add arc $(1,3)$ of length $\ell_{13} = 0$ to the standardized network, which corresponds to add precedence $SS_{13}^{min}(0)$ between activities 1 and 3 (and the related arc to the original network), to force activity 3 to start not before time 0. On the standardized network, the critical path would change to $(1,3,4,5,4,6,7)$ of length 21. If it was $d_3 = 4$, it would be $\ell_{23} = 1$, $\ell_{34} = 4$, and the critical path would not change. Therefore, activity 3 is backward-flexible and forward-inflexible, and, hence, it is also forward-critical. Similarly, if it was $d_5 = 6$, the length of the longest path from node 5 to node 7 would be less than d_5, and hence we need to add arc $(5,7)$ of length $\ell_{57} = d_5$ to force the start time of dummy activity 7 (i.e., the project makespan) to be not less than the finish time of activity 5, which corresponds to add precedence $FS_{57}^{min}(0)$ between activities 5 and 7 (and the related arc to the original network). On the standardized network, the critical path would be $(1,2,3,4,5,7)$ of length 21. Therefore, activity 5 is forward-inflexible, meaning that it is also forward-critical. Finally, activity 4 is both start- and forward-critical, because it is backward-flexible since we can decrease its duration without increasing the length of the critical path $(1,2,3,4,5,4,6,7)$. These results show possible failures of the De Reyck's method.

4 New general results

Referring to the previous analysis, it is possible to prove that:

Proposition 1. *Given a critical activity i such that the longest path in the standardized network from node 1 to node i is equal to 0, if the critical precedence relations ingoing activity i are only of type XF_{hi}^{min} and the critical precedence relations outgoing activity i are of type FX_{ij}^{min} (SX_{ij}^{min}), then activity i is not only finish-critical (backward-critical) as induced by the De Reyck's method but also forward-critical (start-critical).*

Proposition 2. *Given a critical activity i whose duration d_i is equal to the longest path from node i to node n in the standardized network, if the critical precedence relations outgoing activity i are only of type SX_{ij}^{min} and the critical precedence relations ingoing activity i are of type XS_{hi}^{min} (XF_{hi}^{min}), then activity i is not only start-critical (backward-critical) as induced by the De Reyck's method but also forward-critical (finish-critical).*

These results suggest further corrections to the standardized network and consequently to the original network. In particular, in the previous example, if we add arc $(1,3)$ with $\ell_{13}=0$ and arc $(5,7)$ with $\ell_{57}=d_5$ to the standardized network, that correspond to precedence relations $SS_{13}^{min}(0)$ and $FS_{57}^{min}(0)$, respectively, we obtain that by applying the De Reyck's method we identify the correct criticality of activities 3 and 5. However, the criticality of activity 4, involved in the cycle $(4,5,4)$, remains incorrect. The conclusion is that, apart from the corrections, it is necessary a new method to define on a generic project network the right criticality and flexibility of each single activity.

5 Our proposal

Before outline a new method for analyzing activity criticalities and flexibilities, we redefine activity criticalities, independently from the project network representation.

Definition 1. *An activity is* critical *if its earliest and latest start (finish) times are equal.*

Definition 2. *An activity is* start-critical *if it is critical and the project duration increases only if we delay the activity start time.*

This means that, given a start-critical activity, if we maintain fixed its start time and vary (either increase or decrease) its duration, and, hence, vary (either increase or decrease) its finish time, the project duration does not change, meaning that a start-critical activity is bi-flexible. In addition, the finish time of a start-critical activity is not constrained.

Definition 3. *An activity is* finish-critical *if it is critical and the project duration increases only if we delay the activity finish time.*

This means that, given a finish-critical activity, if we maintain fixed its finish time and vary (increase or decrease) its duration, and, hence, vary (increase or decrease) its start time, the project duration does not change, meaning that a finish-critical activity is bi-flexible. In addition, the start time of a finish-critical activity is not constrained.

Definition 4. *An activity is* forward-critical *if it is critical and the project duration increases whether we delay its start time, while maintaining fixed its duration, or we increase its duration while maintaining fixed its start time (apart from project time-infeasibility).*

Therefore, in anyone of the above two cases, also the activity finish time increases, meaning that the forward-criticality *dominates* the finish-criticality. Moreover, apart from the project time-infeasibility, an increase of the duration of a forward-critical activity increases the project duration, meaning that a forward-critical activity is forward-inflexible.

Definition 5. *An activity is* backward-critical *if it is critical and the project duration increases whether we delay its finish time, while maintaining fixed its duration, or we decrease its duration while maintaining fixed its finish time (apart from project time-infeasibility).*

Hence, in anyone of the above two cases, also the activity start time increases, that is, backward-criticality *dominates* start-criticality, and, apart from the project time-infeasibility, a decrease of the duration of a backward-critical activity increases the project duration, that is, a backward-critical activity is backward-inflexible. Our definitions differ from those by De Reyck for which "an activity is forward-critical (backward-critical) if (a) it is start-critical (finish-critical), and (b) when the project duration increases when activity's duration is increased (decreased)" (cfr. p. 124 of Demeulemeester and Herroelen, 2002).

Definition 6. *An activity is* bi-critical *if it is both forward-critical and backward-critical.*

Therefore, a bi-critical activity is bi-inflexible.

We propose the following approach for analyzing activity criticalities and flexibilities, whose correctness is formally proved:

1. Adopt the AON project network representation with minimum time lags.
2. Convert the network into the standardized network (with only $SS_{ij}^{min}(\ell)$ precedences).
3. Correct the standardized network, if necessary, with the addition of new arcs outgoing from source node 1 and/or ingoing to sink node n, also on the basis of Propositions 1 and 2. Consequently, additional precedence relations of type $SS_{1i}^{min}(0)$ outgoing from node 1 and/or of type $FS_{jn}^{min}(0)$ ingoing to node n might have to be considered.
4. Find on the (corrected) standardized network the critical subnetwork composed by all the critical nodes (activities) and all the critical arcs on the standardized network.
5. Trace back the critical nodes and the critical arcs on the original AON project network in order to consider only its critical subnetwork.
6. Determine the types of criticality of each critical activity i on the basis of the precedence types of the couples of critical ingoing and outgoing arcs of i and the existence or not of elementary critical paths passing through these arc couples.
7. Determine possible project time-infeasibility of each critical activity i on the basis of the existence or not of elementary cycles traversing node i on the critical subnetwork.
8. Analyze the flexibility of non-critical activities in order to detect possible project time-infeasibility due to duration changing for these activities.

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